Maxwell's Formulation – Differential Forms on Euclidean Space

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Abstract

One of the greatest advances in theoretical physics of the nineteenth century was Maxwell's formulation of the the equations of electromagnetism. This article uses differential forms to solve a problem related to Maxwell's formulation. The notion of differential form encompasses such ideas as elements of surface area and volume elements, the work exerted by a force, the flow of a fluid, and the curvature of a surface, space or hyperspace. An important operation on differential forms is exterior differentiation, which generalizes the operators div, grad, curl of vector calculus. the study of differential forms, which was initiated by E.Cartan in the years around 1900, is often termed the exterior differential calculus.However, Maxwell's equations have many very important implications in the life of a modern person, so much so that people use devices that function off the principles in Maxwell's equations every day without even knowing it.

1 Introductions to differential forms

1.1 Elementary properties

A *differential form of degree* k or a k*-form* on R^n is an expression

$$\alpha = \sum_I f_I dx_I$$

Here I stands for a *multi-index*$(i_1, i_2, \cdots, i_k)$of *degree* k,that is a "vector" consisting of k integer entries ranging between 1 and n, The f_I are smooth functions on R^n called the *coefficients* of α, and dx_I is an abbreviation for

$$dx_{i_1} dx_{i_2} \cdots dx_{i_k}$$

(The notion $dx_{i_1} \bigwedge dx_{i_2} \bigwedge \cdots \bigwedge dx_{i_k}$ is also often used to distinguish this kind of product from another kind, called the tensor product) For instance the expressions:
$\alpha = \sin x_1 + e^{x_4} dx_1 dx_2 + x_2 x_5^2 dx_2 dx_3 + 6dx_2 dx_4 + \cot x_2 dx_5 dx_3$
$\beta = x_1 x_3 x_5 dx_1 dx_6 dx_3 dx_2$
represent a 2-form on R^5, resp. a 4-form on R^6 [2]. The form α consists of four terms, corresponding to the multi-indices (1,5),(2,3),(2,4),and (5,3), whereas β consists of one term, corresponding to the multi-index(1,6,3,2). Note, however, that α could equally well be regarded as a 2-form on R^6 that does not involve the variable x_6. To avoid such ambiguities it is good practice to state explicitly the domain of definition when writing a differential form [3]. A 0-form on R^n us simply a smooth function (no dx's)

1.2 Exterior derivative

If f is a 0-form, that is a smooth function, we define df to be the 1-form

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Then we have the product or leibniz rule:

$$d(fg) = fdg + gdf$$

If$\alpha = \sum_I f_I dx_I$ is a k-form, each of the coefficient f_I is a smooth function and we define $d\alpha$ to be the k+1 form

$$d\alpha = \sum_I df_I dx_I$$

The operation d is called *exterior differentiation* [1]. An operator of this sort is called a first-order partial differential operator, because it involves the first partial derivatives of the coefficients of a form.

Proposition 1.1. (i)$d(a\alpha + b\beta) = ad\alpha + bd\beta forallk - forms\alpha and\beta andallscalarsaandb.$ (ii)$d(\alpha\beta) = (d\alpha)\beta + (-1)^k \alpha d\beta forallk - form\alpha andl - forms\beta.$

Proposition 1.2. $d(d\alpha) = 0$ for any form α, In short,

$$d^2 = 0$$

1.3 Closed and exact forms and Hodge star operator

A form α is *closed* if $d\alpha = 0$. It is exact if $\alpha = d\beta$ for some form β(of degree one or less).

Proposition 1.3. Every exact form is closed.

Proof: If $\alpha = d(d\beta)$ then $d\alpha = d(d\beta) = 0$ by last section's second proposition. The binomial coefficient C_n^k is the number of ways of selecting k(unordered) objects from a collection of n objects. Equivalently, C_n^k is the number of ways of partitioning a pile of n objects into a pile of k objects and a pile of $n-k$ objects. Thus we see that

$$C_n^k = C_{n-k}^k$$

This means that in a certain sense there are as many $k - forms$. this is a natural way to turn $k - forms$ into $n - k - forms$. This is the *Hodge star operator.* Hodge star of α denoted by $*\alpha$ (or sometimes α^*) and is defined as follows. If $\alpha = \sum_I f_I dx_I$ than

$$*\alpha = \sum_I f_I * dx_I$$

with

$$*dx_I = \varepsilon_I dx_{I^c}$$

Here, for any increasing multi-index I, I^c denote the *complementary* increasing multi-index, which consists of all numbers between 1 and n that do not occur in I. The factor ϵ^I us a sign,

$$varepsilon_I = \begin{cases} 1 & \text{if } dx_I dx_{I^c} = dx_1 dx_2 \cdots dx_n \\ -1 & \text{if } dx_I dx_{I^c} = -dx_1 dx_2 \cdots dx_n \end{cases} \quad (1)$$

In other words, $*dx_I$ is the product of all the $dx_j's$ that do not occur in dx_I, times a factor ± 1 which is chosen in such a way that $dx_I(*dx_I)$ is the volume form:

$$dx_I(*dx_I) = dx_1 dx_2 \cdots dx_n$$

Example. On R^2 we have $dx = dy$ and $dy = -dx$. On R^3 we have

- $*dx = dydz, *(dxdy) = dz,$
- $*dy = -dxdz = dzdx, *(dxdz) = -dy,$
- $*dz = dxdy, *(dydz) = dx.$

This is the reason that 2-forms on R^3 are sometimes written as $fdxdy + gdzdx + hdydz$, in contravention of our rule to write the variables in increasing order. In higher dimensions it is better to stich to the rule.
On R^4 we have

- $*dx_1 = dx_2dx_3dx_4 * dx_3 = dx_1dx_2dx_4$
- $*dx_2 = -dx_1dx_3dx_4 * dx_4 = -dx_1dx_2dx_3$ and
- $*(dx_1dx_2) = dx_3dx_4 * (dx_2dx_3) = dx_1dx_4$
- $*(dx_1dx_3) = -dx_2dx_4 * (dx_2dx_4) = -dx_1dx_3$
- $*(dx_1dx_4) = dx_2dx_3 * (dx_3dx_4) = dx_1dx_2$

1.4 div,grad and curl

A vector field on an open subset Û if R^n is a smooth map $F :\to R^n$. We can write F in components as

$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_n(x) \end{pmatrix}$$

$\alpha = F \cdot dx$,the

$$d * \alpha = d(F \cdot *dx) = divF dx_1 dx_2 \ldots Dx_n$$

Al alternative way of writing this identity is obtained by applying $*$ to both sides, which gives

$$\boxed{divF = *d * \alpha}$$

The correspondence between vector fields and 1-forms behaves in an interesting way with respect to exterior differentiation and the Hodge star operator. For each function f the 1-form $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$ is associated to the vector field

$$F(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i \begin{pmatrix} frac\partial f \partial x_1 \\ frac\partial f \partial x_2 \\ \vdots \\ frac\partial f \partial x_n \end{pmatrix}$$

In three dimensions $*d\alpha$ is a 1-form and so is associated to a vector field, namely

$$curl\mathbf{F} = (\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3})e_1 - (\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3} e_2 + (\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})e_3,$$

the *curl* of $\mathbf{F}$. Thus, for n=3, if $\alpha = \mathbf{F} \cdot dx$, then

$$\boxed{curl\mathbf{F} \cdot dx = *d\alpha.}$$

2 Maxwell's Equations

2.1 Maxwell's Equation

The differential forms of Maxwell's equations as found by Heaviside, while completely valid, are now considered somewhat archaic, and have been replaced by the more useful (equivalent) integral forms. Each law is named according to the person(s) who originally discovered the connections represented by the equation. Here are the four equations:

$$Gauss'slawforelectricity : \oint_{closedsurface} \overrightarrow{E} \cdot d\overrightarrow{A} = \frac{Q_enc}{\epsilon_0}$$

$$Gauss'slawformagnetism : \oint_{closedsurface} \overrightarrow{B} \cdot d\overrightarrow{A} = 0$$

$$Faraday'slaw : \oint \overrightarrow{E} \cdot d\overrightarrow{s} = -\frac{d\emptyset_B}{dt}$$

$$Ampere - Maxwelllaw : \oint \overrightarrow{B} \cdot d\overrightarrow{s} = \mu_0 \epsilon_0 \frac{d\emptyset_E}{dt} + \mu_0 i_{enc}$$

Note:$\oint$ is used to specify a closed loop integral, also known as a line integral. It simply means that in the calculations, we must go all the way around the loop; we cant stop part way through or the equations wont be valid.

2.2 Gauss's law for electricity

Gauss law for electricity, more commonly simply refered to as Gausslaw, states that the closed surface integral of $\overrightarrow{E} \cdot d\overrightarrow{A}$ is equal to the charge enclosed by the surface divided by the electric permittivity of the material the charge is in. Generally, the electric permittivity, denoted ϵ, is taken to be the electric permittivity of free (empty) space, and is written ϵ_0. ($\epsilon_0 \approx 8.85 \cdot 10^12F/m$).

We are free to choose our surfaceits an imaginary construct for the purposes of doing the math, not a real entity. The most common surfaces chosen are spheres and cylinders, because mathematically, symmetry makes applying Gauss law much easier, but theoretically, any closed surface can be chosen and it will give the exact same results.

Imagine a point charge of +**Q** floating in space. Centered around this charge, construct a spherical Gaussian surface of radius R. Since the charge is centered in the sphere, the **E** field points radially outward and has the same magnitude at all points on the sphere. Remember that $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$. Since in this example, $r = R$, this equation becomes $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$

From the definition of electric flux, $\emptyset_E = \oint_{closedsurface} \overrightarrow{E} \cdot d\overrightarrow{A}$, so applying Gauss'law is a way of finding the electric flux through a surface due to a charge $Q \cdot d\overrightarrow{A}$ is a unit vector normal to the surface at all points, and represents a tiny portion of the surface area of the Gaussian surface. The closed surface integral of $d\overrightarrow{A}$ is the surface area, **A**.

Again from the definition of electric flux,

$$\emptyset_E = \oint_{closedsurface} \overrightarrow{E} \cdot d\overrightarrow{A}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$$

$$\emptyset_E = \oint_{closedsurface} (\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}) \cdot d\overrightarrow{A}$$

Since $\overrightarrow{E}$ is pointing radially outward everywhere, it is always parallel to $d\overrightarrow{A}$, and $\overrightarrow{E} \cdot d\overrightarrow{A}$ becomes $(\overrightarrow{E})d\overrightarrow{A}$. Since $\overrightarrow{E}$ is constant at all points on the sphere, it can be moved outside the integral:

$$\emptyset_E = (\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}) \oint_{closedsurface} d\overrightarrow{A}$$

$$\emptyset_E = (\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2})A$$

where **A** is the surface area of the sphere. However, the surface area of a sphere is simply $4\pi R^2$, so this becomes

$$\emptyset_E = (\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2})(4\pi R^2)$$

$$\emptyset_E = \frac{Q}{\epsilon_0}$$

But this, of course, is simply Gauss'law! $\emptyset_E$ is independent of the radius of the sphere, which may seem strange, since $\overrightarrow{E}$ clearly decreases at a rate $\propto 1\backslash R\backslash R^2$; however, since $\overrightarrow{E}$ points away from the charge, no matter how large the radius of the sphere is, the electric field will still penetrate it at some point, and the flux will have to be the same. Mathematically, it works because $\emptyset_E$ is $\overrightarrow{E}$ multiplied by the surface area of the Gaussian surface; $\overrightarrow{E} \propto 1\backslash R\backslash R^2$, and $A \propto R^2$, so their product, $\emptyset_E$ must be independent of R.

Imagine that, instead placing a charge of $+Q$ inside the Gaussian surface, we placed outside. Clearly the electric field still points away from the charge, and at some point, the electric field will pass through the Gaussian surface. On one side of the surface, this will give a negative flux - the electric field is entering the surface! But the electric field will have to leave the Gaussian surface on the other side, creating a positive flux. Since all the field lines that enter the surface must leave again - they don't just stop - the net electric flux will be zero, as predicted by Gauss'law.

Using arguments of symmetry, it is also possible to prove Gauss'law for Gaussian surfaces of other shapes, such as cylinders. It can also be usedin reverse; by dividing both sides of the equation by A after integrating, the electric field caused by various charge configurations can be found for all points in space. An example of this is finding the electric field at all points in space caused by an infinitely large plane of charge density ρ. Its done using a cylindrical

Gaussian surface rather than a spherical one, and while the idea of an infinitely large plane is ridiculous, the results hold true as long as the distance from the plane at which the electric field is being calculated is significantly smaller than the size of the plane, and not near the edge.

2.3 Gauss's law for magnetism

Gauss'law for magnetism is remarkably similar to Gauss law for electricity in form, but means something rather different. Imagine that a magnet was placed in space, and that a spherical Gaussian surface was constructed around it. Remember from the section on magnetism that magnetic fields flow,by convention, from the North pole of a magnet to the South pole. From the definition of magnetic flux, $\emptyset_B = \oint\limits_{closedsurface} \overrightarrow{B} \cdot d\overrightarrow{A}$. Part of the magnetic field will not pierce the Gaussian surface - this portion of the field clearly will not contribute to the flux through the surface, so it can be ignored. The rest of the magnetic field lines will leave through the surface from the North pole of the magnet, but because the field flows from the North pole to the South pole, the same field lines will enter the surface again somewhere on the surface to go to the South pole. Since the flux going out is equal to the flux coming in, the net flux is zero, as indicated by Gauss law for magnetism.

Suppose that instead the magnet was placed outside the Gaussian surface. The same argument applies: any part of the magnetic field that enters the surface will have to leave again through the surface, since it is closed. The positive flux will equal the negative flux, theyll cancel, and the net flux will be zero. Again, this matches what was predicted by Gauss law.

Pretend that a special magnet with only a North pole, and no South pole, existed. This would be called a magnetic monopole. All the magnetic field lines would point away from this theoretical magnetic monopole, just like the electric field lines point away from a positive charge Q. If a Gaussian surface was constructed around this monopole, there would obviously be a positive flux going through the surface, because the magnetic field is leaving, and it isnt coming back in! Gauss law for magnetism, however, very clearly says that the flux should be zero! This means that according to Gauss, there can be no magnetic monopoles - all magnets must have two poles. Although some people are looking for magnetic monopoles, none have ever been observed, and if one is ever found, it will mean that Gauss law for magnetism is incorrect.

2.4 Faraday's law

According to the definition of magnetic flux, $\emptyset_B$, a magnetic field passing through an area A will create magnetic flux. Imagine that a circular loop of wire of radius R is placed in a magnetic field $\overrightarrow{B}$, perpendicular to the direction of the field. The flux through the loop is clearly the strength of the magnetic field multiplied by the area of the loop:$\emptyset_B = \overrightarrow{B}(\pi R^2)$. Now imagine that the magnetic field began changing with time at a rate of $\frac{d\overrightarrow{B}}{dt}$. The change in flux with time would be $\frac{d\emptyset_B}{dt} = (\pi R^2)\frac{d\overrightarrow{B}}{dt}$. The flux could also be changed by altering the area of the loop, but since changing the area of the loop in real applications is not as practical as changing the magnetic field, and since the mathematics are largely similar, only the case of changing magnetic fields will be examined. As was observed by Faraday, when $\emptyset_B$ through the loop is changing, a voltage is induced in the loop in an attempt by the system to "fight" the change. A current will then flow in the loop as determined by the Ohm'law, $V = IR$, where R is the resistance of the loop.

Consider again the scenario above. Faraday's law contains the integral of $\overrightarrow{E} \cdot d\overrightarrow{s}$. The $d\overrightarrow{s}$ represents an infinitely small portion of the loop of wire. Recall that an electric field multiplied by a distance represents a voltage. We can go around the loop in either direction and it won't affect our results other than a change in sign - but that change in sign is to be expected, because in one direction, we would be increasing in potential as we went around, and in the other direction, we would be decreasing in potential! From Faraday'law, we have

$$\oint \overrightarrow{E} \cdot d\overrightarrow{s} = -\frac{d\emptyset_B}{dt}$$

2.5 Ampere-Maxwell law

Ampere observed that current flowing through a wire created a magnetic field around the wire, and formulated the equation

$$\oint \overrightarrow{B} \cdot d\overrightarrow{s} = \mu_0 i_{enc}$$

i^enc, meaning current enclosed, is perhaps a deceptive notation. Currentcan not be enclosed; rather, what is meant is the current that passes through the interior of the closed loop. 0is a constant called the magnetic permeability of free space; if there is a material present instead of simply space, μ_0 is replaced with μ for the material.

Ampere's law is used by simply selecting any closed loop, traversing it with small elements $d\overrightarrow{s}$, and solving the resulting equation. It is key to note that any closed loop can be selected a flat disc, or perhaps a shape more similar to a grocery bag and it will give the same results.

Ampere's law predicted the magnetic field very accurately, but Maxwell noticed that there was a piece missing. He noted that a capacitor is made of two conducting plates separated by some distance d, and that while the capacitor was charging, positive charge accumulated on one plate, and negative charge accumulated on the other plate, but that no current passed between the plates. A capacitor is essentially a gap in a circuit, but because of its nature, the circuit is still complete. However, using Amperes law to find the magnetic field at a point in space, it was possible to select one closed loop passing through the capacitor, so that no current passed through the closed loop. This would indicate that there was no magnetic field at that point. However, another closed loop could be selected for the same point that passed through one of the wires connected to the capacitor the law leaves us free to choose our own closed loop and since current flows in the wire, the law would clearly indicate that there was a magnetic field at that point! Clearly this could not be, so something had to be missing.

Maxwell named the missing term displacement current, even though it is not really a current at all, but rather is the changing electric field within the capacitor. Since charge is accumulating on the plates of the capacitor, there is a changing electric field between the two plates. By introducing the term $\mu_0\epsilon 0\frac{dphie}{dt}$, Maxwell completed the equation, now called the Ampere-Maxwell law:

$$\oint \overrightarrow{B} \cdot d\overrightarrow{s} = \mu_0\epsilon_0\frac{d\emptyset_E}{dt} + \mu_0 i_{enc}$$

When there is no changing electric field, $fracd\emptyset_E dt = 0$ and the law simply becomes Amperes law.

3 Differential forms and Maxwell's equation

3.1 Relationship

We denoted E:

$$E = E_x dx + E_y dy + E_z dz$$

The electric field is a one-form because its duality product with a vector is s scalar. An example of a two-from is the density of electric current $\mathbf{J}$ which is integrable over a surface,

$$\mathbf{J} = J_{xy}dx \bigwedge dy + J_{yz}dy \bigwedge dz + J_{zx}dz \bigwedge dx$$

The surface is defined by a two-dimensional manifold S which can be approximated by a chain of two-simplexes which are bi-vectors

$$\Delta\mathbf{S}_{ij} = (1/2)\Delta\mathbf{r}_i \bigwedge \Delta\mathbf{r}_j.$$

he current $\mathbf{I}$ through the surface is expressed as the integral

$$I = \lim \sum \mathbf{J} \mid \Delta\mathbf{S}_{ij} = \mathbf{J} \mid \S$$

Finally, as an example of a zero-form which is not integrable over a space region is the scalar potential scalar $\emptyset$. To summarize, various basic electromagnetic quantities can be expressed in **3D** Euclidean differential forms as follows:

- **Zero-forms:**scalar potential $\emptyset$, magnetic scalar potential $\emptyset_m$;
- **One-forms:**electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, vector potential $\mathbf{A}$, magnetic vector potential $\mathbf{A}_m$;
- **Two-forms:**electric flux density $\mathbf{D}$, magnetic flux density $\mathbf{B}$, electric current density $\mathbf{J}$, magnetic current density $\mathbf{J}_m$;
- **Three-forms:**electric charge density ϱ ;

4 Solution of Maxwell's Equations

4.1 Maxwell's Equations

Maxwell's formulation of the equations of electromagnetism:

$$curlE = -\frac{1}{c}\frac{\partial B}{\partial t}(1)$$

$$curlH = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial D}{\partial t}(2)$$

$$divD = 4\pi\rho(3)$$

$$divB = 0(4)$$

Here c is the speed of light, E is the electirc field, H is the magnetic field, J it the density of electic current, ρ is the density of electic charge, B is the magnetic induction and D is the dielectric displacement. E, H, B, J and D are vector fields and ρ is a fuction on R^3 and all depend o time t. $\alpha = (E_1dx_1 + E_2dx_2 + E_3dx_3)dx_4 + B_1dx_2dx_3 + B_2dx_3dx_1 + B_3dx_1dx_2(5)$ $\beta = -(H_1dx_1+H_2dx_2+H_3dx_3)dx_4+D_1dx_2dx_3+D_2dx_3dx_1+D_3dx_1dx_2(6)$ $\gamma = \frac{1}{c}(J_1dx_2dx_3+J_2dx_3dx_1+J_3dx_1dx_2)dx_4 - \rho dx_1dx_2dx_3(7)$

Problem 4.1. show that Maxwell's equations are equivalent to

$$d\alpha = 0$$

$$d\beta + 4\pi\gamma = 0$$

Answer: From (5), we know
$(1)d\alpha = \frac{\partial E_1}{\partial x_2}dx_2dx_1 + \frac{\partial E_1}{\partial x_3}dx_3dx_1 + \frac{\partial E_2}{\partial x_1}dx_1dx_2 + \frac{\partial E_2}{\partial x_3}dx_3dx_2 + \frac{\partial E_3}{\partial x_1}dx_1dx_3 + \frac{\partial E_3}{\partial x_2}dx_2dx_3)dx_4 + \frac{\partial B_1}{\partial x_1}dx_1dx_2dx_3\frac{\partial B_1}{\partial t}dtdx_2dx_3 + \frac{\partial B_2}{\partial x_2}dx_2dx_3dx_1 + \frac{\partial B_2}{\partial t}dtdx_3dx_1 + \frac{\partial B_3}{\partial x_3}dx_3dx_1dx_2 + \frac{\partial B_3}{\partial t}dtdx_1dx_2$
$= (-\frac{\partial E_1}{\partial x_2}\cdot c + \frac{\partial E_2}{\partial x_1}\cdot c + \frac{\partial B_3}{\partial t})dx_1dx_2dt + (-\frac{\partial E_1}{\partial x_3}\cdot c + \frac{\partial E_3}{\partial x_1}\cdot c + \frac{\partial B_2}{\partial t})dx_1dx_3dt + (-\frac{\partial E_2}{\partial x_3}\cdot c + \frac{\partial E_3}{\partial x_2}\cdot c + \frac{\partial B_1}{\partial t})dx_2dx_3dt + (\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3})dx_1dx_2dx_3$
$= 0$
$\Longleftrightarrow$
$-\frac{\partial E_1}{\partial x_2}\cdot c + \frac{\partial E_2}{\partial x_1}\cdot c + \frac{\partial B_3}{\partial t} = 0$
$-\frac{\partial E_1}{\partial x_3}\cdot c + \frac{\partial E_3}{\partial x_1}\cdot c + \frac{\partial B_2}{\partial t} = 0$
$-\frac{\partial E_2}{\partial x_3}\cdot c + \frac{\partial E_3}{\partial x_2}\cdot c + \frac{\partial B_1}{\partial t} = 0$
$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0$
$\Longleftrightarrow$
$curlE = -\frac{1}{c}\frac{\partial B}{\partial t}$
$divB = 0$
(2)Using the same method as above, we get
$d\beta = -(-\frac{\partial H_1}{\partial x_2}\cdot c + \frac{\partial H_2}{\partial x_1}\cdot c - \frac{\partial D_3}{\partial t})dx_1dx_2dt - (-\frac{\partial H_1}{\partial x_3}\cdot c + \frac{\partial H_3}{\partial x_1}\cdot c - \frac{\partial D_2}{\partial t})dx_1dx_3dt - (-\frac{\partial H_2}{\partial x_3}\cdot c + \frac{\partial H_3}{\partial x_2}\cdot c - \frac{\partial D_1}{\partial t})dx_2dx_3dt + (\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3})dx_1dx_2dx_3$
$d\beta + 4\pi\gamma = d\beta + 4\pi(\frac{1}{c}(J_1dx_2dx_3 + J_2dx_3dx_1 + J_3dx_1dx_2)dx_4 - \rho dx_1dx_2dx_3) = 0$
$\Longleftrightarrow$
$-4\pi\rho + \frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3} = 0$
$-\frac{\partial H_2}{\partial x_3}\cdot c + \frac{\partial H_3}{\partial x_2}\cdot c - \frac{\partial D_1}{\partial t} - 4\pi J_1 = 0$
$-\frac{\partial H_1}{\partial x_3}\cdot c + \frac{\partial H_3}{\partial x_1}\cdot c - \frac{\partial D_2}{\partial t} - 4\pi J_2 = 0$
$-\frac{\partial H_1}{\partial x_2}\cdot c + \frac{\partial H_2}{\partial x_1}\cdot c - \frac{\partial D_3}{\partial t} - 4\pi J_3 = 0$
$\Longleftrightarrow$
$curlH = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial D}{\partial t}$
$divD = 4\pi\rho$

Problem 4.2. Conclude that γ is closed and that $divJ + \partial\rho/\partial t = 0$

Answer: Form equation (7)
$d\gamma = \frac{1}{c}(\frac{\partial J_1}{\partial x_1}dx_1dx_2dx_3 + \frac{\partial J_2}{\partial x_2}dx_1dx_2dx_3 + \frac{\partial J_3}{\partial x_3}dx_1dx_2dx_3)dx_4 + \frac{\partial \rho}{\partial t}dx_1dx_2dx_3dt$
When γ is closed, we know the former formula: =0

that means:
$\frac{1}{c}(\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3}) + \frac{\partial \rho}{\partial t} = 0$
$\Longleftrightarrow$
$divJ + \partial\rho/\partial t = 0$

Problem 4.3. In vacuum one has $E = D$ and $H = B$. show that in vacuum $\beta = *\alpha$, Use the relative Hodge star of α in EX.2.17

Answer: From equation (5)
$*\alpha = E_1dx_2dx_3 - E_2dx_1dx_3 + E_3dx_1dx_2 - (B_1dx_1dx_4 + B_2dx_2dx_4 + B_3dx_3dx_4)(9)$
$sinceE = D \ and \ H = B$
$*\alpha = D_1dx_2dx_3 - D_2dx_1dx_3 + D_3dx_1dx_2 - (H_1dx_1dx_4 + H_2dx_2dx_4 + H_3dx_3dx_4)$
$= \beta$

Problem 4.4. Free space is a vacuum without charges or currents. Show that the Maxwell's equations in free space are equivalent to $d\alpha = d * \alpha = 0$

Answer: Since free space is a vacuum without charges or currents then $J = 0$ and $\rho = 0$, E=D and H=B
From the first problem we know that $d\alpha = \Longleftrightarrow$ equation (1) and (4).
and we only need to get that $d * \alpha = 0 \Longleftrightarrow d\beta + 4\pi\gamma = 0$
since $J = 0$ and $\rho = 0$, we know $f = 0$
that is $d\beta = 0 \Longleftrightarrow d * \alpha = 0$ (from in vacuum $\beta = *\alpha$)
problem was done.

Problem 4.5. Let $f, g : R \longrightarrow R$ be any smooth functions and define

$$E(x) = \begin{matrix} 0 \\ f(x_1 - x_4) \\ g(x_1 - x_4) \end{matrix} \quad (2)$$

$$B(x) = \begin{matrix} 0 \\ -g(x_1 - x_4) \\ f(x_1 - x_4) \end{matrix} \quad (3)$$

Show that the corresponding 2-form α satisfies the free Maxwell's equations $d\alpha = d * \alpha = 0$. Such serious are called *electromagnetic waves*. Explain why. In what direction do these waves travel.

Answer: Since $f(x_1 - x_4)$, we know $\frac{\partial f}{\partial x_1} = -\frac{\partial f}{\partial x_4}(*)$
The same thing happens to $g(x_1 - x_4)$, that is $\frac{\partial g}{\partial x_1} = -\frac{\partial g}{\partial x_4}(**)$
Bring E(x) and B(x) to $d\alpha$
We get $d\alpha = (\frac{\partial E_3}{\partial x_1} - \frac{\partial B_2}{\partial x_4}dx_1dx_3dx_4 + (\frac{\partial E_2}{\partial x_1} + \frac{\partial B_3}{\partial x_4}dx_1dx_2dx_4$
From (*) and (**), we know $d\alpha$=0
The same thing happens to $d * \alpha$ and $d * \alpha = 0$
The wave travels in the direction of x-axis, because for E and B the 1st dimension part are 0; At the same time, the 2nd and 3rd dimension is changed only related to x_1 and time t.

References

[1] Differential Forms in Electromagnetics, Ismo V. Lindell, Wiley-IEEE Press, April 2004.

[2] Introduction to Differential Forms, Donu Arapura, March 2010.

[3] Maxwell's Equations, Matt Hansen, 2004.